

ARITHMETIC PROPERTIES FOR APÉRY-LIKE NUMBERS

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ABSTRACT. It is known that the numbers which occur in Apéry's proof of the irrationality of $\zeta(2)$ have many interesting congruence properties while the associated generating function satisfies a second order differential equation. We prove congruences for numbers which arise in Beukers' and Zagier's study of integral solutions of Apéry-like differential equations.

1. INTRODUCTION

In the course of his work on proving the irrationality of $\zeta(2)$, Apéry introduced, for a positive integer n , the following sequence of numbers [3], [25]

$$B(n) := \sum_{j=0}^n \binom{n+j}{j} \binom{n}{j}^2.$$

Several authors [4], [10], [11] have subsequently studied many interesting congruence properties for $B(n)$ and its generalizations. In [23], the authors were able to relate the $B(n)$'s to the p -th Fourier coefficient of a modular form. If we define

$$\eta^6(4z) := \sum_{n=1}^{\infty} a(n)q^n$$

where

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function, $q := e^{2\pi iz}$ and $z \in \mathbb{H}$, then Beukers and Steinstra proved using the formal Brauer group of some elliptic K3-surfaces that for all odd primes p and any nonnegative integers m, r with m odd, we have

$$(1) \quad B\left(\frac{mp^r - 1}{2}\right) - a(p)B\left(\frac{mp^{r-1} - 1}{2}\right) + (-1)^{\frac{p-1}{2}} p^2 B\left(\frac{mp^{r-2} - 1}{2}\right) \equiv 0 \pmod{p^r}.$$

They also conjectured the congruence

$$(2) \quad B\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

which does not follow from (1) and has been proven in [1] and [26]. Congruence (2) is but one example of a general phenomena called *Supercongruences*. This term appeared in [4] and was the

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subject of the Ph.D. thesis of Coster [9]. It refers to the fact that congruences of this type are stronger than the ones suggested by formal group theory. Other examples of supercongruences have been observed in the context of number theory [13], [16], [17], [19], [30], mathematical physics [14], [15], and algebraic geometry [7], [21]. Currently, there is no systematic explanation for such congruences. Perhaps, as mentioned in [23], they are related to formal Chow groups [22].

It is known that the $B(n)$'s satisfy the recurrence relation

$$(n+1)^2 B(n+1) = (11n^2 + 11n + 3)B(n) + n^2 B(n-1)$$

for $n \geq 1$. This implies that the generating function

$$\mathcal{B}(t) = \sum_{n=0}^{\infty} B(n)t^n$$

satisfies the differential equation

$$L\mathcal{B}(t) = 0$$

where

$$L = t(t^2 + 11t - 1)\frac{d^2}{dt^2} + (3t^2 + 22t - 1)\frac{d}{dt} + t - 3.$$

In [6], Beukers studies the differential equation

$$(3) \quad ((t^3 + at^2 + bt)F'(t))' + (t - \lambda)F(t) = 0$$

where a, b and λ are rational parameters and asks for which values of these parameters this equation has a solution in $\mathbb{Z}[[t]]$. This equation has a unique solution which is regular at the origin with $F(0) = 0$ given by

$$F(t) = \sum_{n=0}^{\infty} u(n)t^n$$

with $u(0) = 1$ and satisfies the recurrence relation

$$b(n+1)^2 u(n+1) + (an^2 + an - \lambda)u(n) + n^2 u(n-1) = 0$$

where $n \geq 1$. In [29], Zagier describes a search over a suitably chosen domain of 100 million triples (a, b, λ) . He finds 36 triples which yield an integral solution to (3). Of these, he classifies six as “sporadic” cases and conjectures that, up to normalization, these are the only cases with $a \neq 0$, $a^2 \neq 4b$ where (3) has an integral solution. All six cases (which include $B(n)$) have a binomial sum representation and a geometric origin (see [2], [29]).

The purpose of this brief note is to prove congruences, akin to (1) and (2), in three of the “sporadic” cases which can be expressed in terms of binomial sums and have a parametrization in terms of modular functions. The remaining cases can be done similarly. Let

$$A(n) := \sum_{k=0}^n \binom{n}{k}^3,$$

$$C(n) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k},$$

and

$$E(n) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 4^{n-2k} \binom{n}{2k} \binom{2k}{k}^2.$$

Our main result is the following.

Theorem 1.1. *Let*

$$(4) \quad -\frac{1}{9} \frac{\eta(z)^9}{\eta(3z)^3} - \frac{8}{9} \frac{\eta(2z)^9}{\eta(6z)^3} := \sum_{n=0}^{\infty} \gamma(n) q^n$$

and

$$(5) \quad -\frac{1}{4} \frac{\eta(2z)^{30}}{\eta(z)^{12} \eta(4z)^{12}} + 4 \frac{\eta(4z)^4 \eta(2z)^6}{\eta(z)^4} := \sum_{n=0}^{\infty} \lambda(n) q^n.$$

Then for primes $p \geq 5$ and integers $m, r \geq 1$, we have

$$(6) \quad A(mp^r) - \gamma(p)A(mp^{r-1}) + \left(\frac{-3}{p}\right)p^3 A(mp^{r-2}) \equiv 0 \pmod{p^r},$$

$$(7) \quad C(mp^r) - \gamma(p)C(mp^{r-1}) + \left(\frac{-3}{p}\right)p^3 C(mp^{r-2}) \equiv 0 \pmod{p^r},$$

and

$$(8) \quad E(mp^r) - \lambda(p)E(mp^{r-1}) + \left(\frac{-4}{p}\right)p^3 E(mp^{r-2}) \equiv 0 \pmod{p^r}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p .

Using Theorem 1.1, we deduce the following.

Corollary 1.2. *Let $p \geq 5$ be a prime. We have*

$$(9) \quad A(p) \equiv 2\gamma(p) \pmod{p^2},$$

$$(10) \quad C(p) \equiv 3\gamma(p) \pmod{p^2},$$

and

$$(11) \quad E(p) \equiv 4\lambda(p) \pmod{p^2}.$$

The proof for each of the congruences in Corollary 1.2 is simpler than that of (2) as it only requires a short combinatorial argument and the fact that $\gamma(p)$ and $\lambda(p)$ are coefficients of Hecke eigenforms. To prove (2), one uses Greene's notion of Gaussian hypergeometric series (see Chapter 11 of [20]). Finally, there exist more congruences for the numbers $A(n)$ and $C(n)$ of the form (1) which involve the p -th Fourier coefficients of $\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$ and $\eta^3(6z)\eta^3(2z)$ respectively; see [23]. For supercongruences concerning these modular forms, see [18].

The paper is organized as follows. In Section 2 we recall some properties of integral weight modular forms and prove a congruence for an alternating harmonic sum. The proofs of Theorem 1.1 and Corollary 1.2 are given in Section 3.

2. PRELIMINARIES

We first briefly recall some basic facts concerning integer weight modular forms (see [20]). Let $M_k(\Gamma_0(N), \chi)$ be the space of modular forms of weight k on $\Gamma_0(N)$ with character χ . For each prime p we have the Hecke operator $T_{p, \chi}$ whose action on $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ is given by

$$f(z) | T_{p, \chi} = \sum_{n=0}^{\infty} \left(a(np) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right) \right) q^n,$$

where $a\left(\frac{n}{p}\right) = 0$ if $p \nmid n$. If $f(z)$ is a normalized eigenform (so $a(1) = 1$) for all of the $T_{p, \chi}$ such that $a(0) \neq 0$, then

$$(12) \quad a(p) = 1 + \chi(p)p^{k-1}.$$

A function of the form

$$f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$$

where $N \geq 1$ and $r_\delta \in \mathbb{Z}$ is called an *eta-quotient*. If $f(z)$ is an eta-quotient with

$$k := \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z},$$

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},$$

$$N \sum_{\delta|N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24},$$

and $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$ where

$$\chi(d) := \left(\frac{(-1)^k s}{d} \right), \quad s := \prod_{\delta|N} \delta^{r_\delta}.$$

Since $\eta(z)$ is analytic and does not vanish on \mathbb{H} , $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$ if the associated orders of vanishing are nonnegative. If c and d are positive integers with $\gcd(c, d) = 1$ and $d \mid N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{N}{24d \gcd(d, \frac{N}{d})} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\delta}.$$

We also need a folklore result. If $t(z)$ is a modular function with respect to a discrete subgroup of $SL_2(\mathbb{R})$, commensurable with $SL_2(\mathbb{Z})$, and F is modular form of weight k with respect to the same group, then F as a function of t satisfies a linear differential equation of order $k+1$ with coefficients algebraic in t . This result appeared in [24] and recently Yang [28] has given an elegant proof whereby the coefficients in the differential equation can be explicitly computed. For $k=1$, this equation is of the form

$$D_t^2 F + p_1(t) D_t F + p_2(t) F = 0$$

where

$$\begin{aligned} D_t &:= t \frac{d}{dt}, & D_q &:= q \frac{d}{dq}, \\ G_1 &:= \frac{D_q t}{t}, & G_2 &:= \frac{D_q F}{F}, \end{aligned}$$

and

$$p_1(t) := \frac{D_q G_1 - 2G_1 G_2/k}{G_1^2}, \quad p_2(t) := -\frac{D_q G_2 - G_2^2/k}{G_1^2}.$$

We now recall the following result, due to Verrill [27], which generalizes Theorem 4 in [5].

Theorem 2.1. *Let $t(z)$ be a modular function for $\Gamma_0(N)$, $f(z) \in M_k(\Gamma_0(N), \chi)$, and $L \in M_{k+2}(\Gamma_0(N), \chi)$ be a Hecke eigenform. Write*

$$f(t) = \sum_{n=1}^{\infty} c(n) t^n, \quad L(q) = \sum_{n=1}^{\infty} \gamma(n) q^n.$$

If for some integers a_d , $d|M$ for some integer M , we have the following equality

$$f(q) \frac{q \frac{dt}{dq}}{t} = \sum_{d|M} a_d L(q^d),$$

then for primes $p \nmid NM$, and integers $m, r \geq 1$, we have

$$c(mp^r) - \gamma(p)c(mp^{r-1}) + \chi(p)p^{k+1}c(mp^{r-2}) \equiv 0 \pmod{p^r}.$$

Finally, we require a congruence concerning an alternating harmonic sum. Recall that for $n \in \mathbb{N}$, the harmonic sum is defined by

$$H_n := \sum_{k=1}^n \frac{1}{k}.$$

Proposition 2.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \binom{-1/2}{k}^2 \equiv -4 \sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \pmod{p}.$$

Proof. Noting that $\binom{u+k}{k} = (-1)^k \binom{-1-u}{k}$, we get

$$(13) \quad \binom{\frac{p-1}{2} + k}{k} \binom{\frac{p-1}{2}}{k} = (-1)^k \binom{-\frac{1}{2} - \frac{p}{2}}{k} \binom{-\frac{1}{2} + \frac{p}{2}}{k} \equiv (-1)^k \binom{-\frac{1}{2}}{k}^2 \pmod{p}.$$

By (13) and the fact that

$$H_{\frac{p-1}{2}} = H_{p-1} + \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \equiv 2 \sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \pmod{p},$$

the result follows upon taking $n = \frac{p-1}{2}$ in the identity

$$\sum_{k=1}^n \frac{(-1)^k}{k} \binom{n+k}{k} \binom{n}{k} = -2H_n$$

and reducing modulo p . □

3. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

Proof of Theorem 1.1. We consider the modular function for $\Gamma_0(6)$

$$t(z) = \frac{\eta(z)^3 \eta(6z)^9}{\eta(2z)^3 \eta(3z)^9}$$

and claim that the equality

$$(14) \quad f(t) := \sum_{n=0}^{\infty} A(n) t^n = \frac{\eta(2z) \eta(3z)^6}{\eta(z)^2 \eta(6z)^3}$$

holds. In order to see this, note that (as $A(n)$ is #5 on Zagier's list) $f(t)$ satisfies the differential equation

$$(15) \quad \left(t^3 + \frac{7}{8}t^2 - \frac{1}{8}t\right) f''(t) + \left(3t^2 + \frac{7}{4}t - \frac{1}{8}\right) f'(t) + \left(t + \frac{1}{4}\right) f(t) = 0.$$

Let $F = \frac{\eta(2z) \eta(3z)^6}{\eta(z)^2 \eta(6z)^3}$. The function field of modular functions on $\Gamma_0(6)$ is generated by

$$g = \frac{\eta(z)^4 \eta(6z)^8}{\eta(2z)^8 \eta(3z)^4}$$

and the values at the cusps are given by

$$g(i\infty) = 0, \quad g(0) = 1/9, \quad g(1/2) = \infty, \quad g(1/3) = 1.$$

The modular functions t and g satisfy the relation

$$(16) \quad t = \frac{g}{1-g}.$$

Let $G(z)$ be the Eisenstein series of weight 2 on $SL_2(\mathbb{Z})$. Then the Eisenstein series of weight 2 associated with the cusps of $\Gamma_0(6)$ are given by

$$\begin{aligned} P_\infty &= \frac{1}{24}(G(z) - 4G(2z) - 9G(3z) + 36G(6z)), \\ P_0 &= \frac{3}{2}(G(z) - G(2z) - G(3z) + 6G(6z)), \\ P_{1/2} &= -\frac{3}{8}(G(z) - 4G(2z) - G(3z) + 4G(6z)), \\ P_{1/3} &= -\frac{1}{6}(G(z) - G(2z) - 9G(3z) + 9G(6z)). \end{aligned}$$

The value of the Eisenstein series P_a at the cusp a is 1 and at other cusps is 0. We find that

$$G_1 := \frac{D_q t}{t} = P_\infty - 1/2P_{1/3}$$

and

$$G_2 := \frac{D_q F}{F} = 1/2P_{1/3}.$$

We now let

$$A_2 = G(z) - 2G(2z), \quad A_3 = G(z) - 3G(3z), \quad A_6 = G(z) - 6G(6z).$$

Comparing the first few terms of $D_q G_1 - 2G_1 G_2$ with those of A_2^2 , A_3^2 , A_6^2 , $A_2 A_3$ and $A_2 A_6$, we have

$$(17) \quad D_q G_1 - 2G_1 G_2 = \frac{1}{384}(-29A_2^2 + 51A_3^2 + 3A_6^2 + 20A_2 A_3 - 58A_2 A_6)$$

and

$$(18) \quad D_q G_2 - G_2^2 = \frac{1}{48}(-2A_2^2 + 3A_3^2 - 2A_2 A_6).$$

We now express $(D_q G_1 - 2G_1 G_2)/G_1^2$ and $(D_q G_2 - G_2^2)/G_1^2$ in terms of t . Our first step is to find expressions for $(D_q G_1 - 2G_1 G_2)/P^2$, G_1^2/P^2 and $(D_q G_2 - G_2^2)/P^2$ where

$$P := \frac{1}{3}P_{1/2} - \frac{1}{2}P_{1/3}$$

is a modular form of weight 2 with zeros at ∞ and 0 and no zeros elsewhere. To express G_1/P in term of g , we note that G_1/P is of the form

$$\frac{a_0 + a_1 g + a_2 g^2}{b_0 + b_1 g + b_2 g^2}$$

where $b_0 + b_1 g + b_2 g^2 = (g - g(i\infty))(g - g(0)) = g(g - 1/9)$. We observe that G_1/P starts from q^{-1} and

$$G_1(0) = 0, \quad G_1(1/2) = 0, \quad G_1(\infty) = 1, \quad G_1(1/3) = 1/2.$$

This implies that $G_1/P = 1/g$. Similarly, we have

$$\begin{aligned}\frac{A_2}{P} &= \frac{-1 - 18g + 27g^2}{g(1 - 9g)}, \\ \frac{A_3}{P} &= \frac{-2 - 12g + 18g^2}{g(1 - 9g)}, \\ \frac{A_6}{P} &= \frac{-5 + 6g - 9g^2}{g(1 - 9g)}.\end{aligned}$$

Substituting these expressions into (17) and (18), we have

$$\frac{D_q G_1 - 2G_1 G_2}{G_1^2} = \frac{g(-7 + 54g + 81g^2)}{1 - 18g + 81g^2}$$

and

$$\frac{D_q G_2 - G_2^2}{G_1^2} = \frac{-2g + 12g^2 + 54g^3}{1 - 18g + 81g^2}.$$

By (16),

$$p_1(t) := \frac{D_q G_1 - 2G_1 G_2}{G_1^2} = \frac{-7t + 40t^2 + 128t^3}{1 - 15t + 48t^2 + 64t^3}$$

and

$$p_2(t) := \frac{D_q G_2 - G_2^2}{G_1^2} = \frac{-2t + 8t^2 + 64t^3}{1 - 15t + 48t^2 + 64t^3}.$$

By the folklore result and some simplification, F as a function of t satisfies the differential equation

$$F'' + q_1(t)F' + q_2(t)F = 0$$

where

$$q_1(t) := \frac{p_1(t) + 1}{t} = \frac{1 - 22t + 88t^2 + 192t^3}{t(1 - 15t + 48t^2 + 64t^3)}$$

and

$$q_2(t) := \frac{-2 + 8t + 64t^2}{t(1 - 15t + 48t^2 + 64t^3)}.$$

This is equivalent to (15) and thus by uniqueness, (14) follows. We now take $L(q)$ to be equal to (4) as it is a Hecke eigenform in $M_3(\Gamma_0(6), (\frac{-3}{d}))$ (one need only use the results on eta-quotients in Section 2 and check a finite number of primes [12]) and note that

$$f(q) \frac{q \frac{dt}{dq}}{t} = L(q).$$

An application of Theorem 2.1 then yields (6). If we now take the modular function for $\Gamma_0(6)$

$$t(z) = g = \frac{\eta(z)^4 \eta(6z)^8}{\eta(2z)^8 \eta(3z)^4},$$

then we claim that

$$(19) \quad f(t) := \sum_{n=0}^{\infty} C(n)t^n = \frac{\eta(2z)^6 \eta(3z)}{\eta(z)^3 \eta(6z)^2}.$$

We have that $f(t)$ satisfies ($C(n)$ is #8 on Zagier's list)

$$(20) \quad \left(t^3 - \frac{9}{10}t^2 + \frac{1}{9}t\right)f''(t) + \left(3t^2 - \frac{9}{5}t + \frac{1}{9}\right)f'(t) + \left(t - \frac{1}{3}\right)f(t) = 0.$$

Let $F = \frac{\eta(2z)^6 \eta(3z)}{\eta(z)^3 \eta(6z)^2}$. In this case, we have

$$G_1 := \frac{D_q(t)}{t} = P_{\infty} - 1/3P_{1/2}$$

and

$$G_2 := \frac{D_q(F)}{F} = 1/3P_{1/2}.$$

Comparing the first few terms of $D_q G_1 - 2G_1 G_2$ with those of A_2^2 , A_3^2 , A_6^2 , $A_2 A_3$ and $A_2 A_6$, we have

$$(21) \quad D_q G_1 - 2G_1 G_2 = \frac{1}{288}(57A_2^2 - 3A_3^2 + 9A_6^2 + 20A_2 A_3 - 62A_2 A_6)$$

and

$$(22) \quad D_q G_2 - G_2^2 = \frac{1}{96}(9A_2^2 - A_3^2 + A_6^2 - 6A_2 A_6).$$

Proceeding as in the first case, we get $G_1/P = (1-g)/g$. Substituting values of G_1/P , A_2/P , A_3/P , A_6/P into (21) and (22), we have

$$p_1(t) := \frac{D_q G_1 - 2G_1 G_2}{G_1^2} = \frac{-10t + 118t^2 - 270t^3 + 162t^4}{(1-t)^2(1-9t)^2}$$

and

$$p_2(t) := \frac{D_q G_2 - G_2^2}{G_1^2} = \frac{-3t + 39t^2 - 117t^3 + 81t^4}{(1-t)^2(1-9t)^2}.$$

By the folklore result and simplification, F as a function of t satisfies the differential equation

$$F'' + q_1(t)F' + q_2(t)F = 0$$

where

$$q_1(t) := \frac{p_1(t) + 1}{t} = \frac{1 - 30t + 236t^2 - 450t^3 + 243t^4}{t(1-t)^2(1-9t)^2}$$

and

$$q_2(t) := \frac{-3t + 39t^2 - 117t^3 + 81t^4}{(1-t)^2(1-9t)^2}.$$

This is equivalent to (20) and thus by uniqueness, (19) follows. If we take $L(q)$ to be (4), then

$$f(q) \frac{q \frac{dt}{dq}}{t} = L(q).$$

Applying Theorem 2.1 yields (7). As the proof of (8) is similar, we only mention that if we choose the modular function on $\Gamma_0(8)$

$$t(z) = \frac{\eta(z)^4 \eta(4z)^2 \eta(8z)^4}{\eta(2z)^{10}},$$

then

$$f(t) := \sum_{n=0}^{\infty} E(n)t^n = \frac{\eta(2z)^{10}}{\eta(z)^4 \eta(4z)^4}$$

satisfies ($E(n)$ is #10 on Zagier's list)

$$\left(t^3 - \frac{3}{8}t^2 + \frac{1}{32}t\right)f''(t) + \left(3t^2 - \frac{3}{4}t + \frac{1}{32}\right)f'(t) + \left(t - \frac{1}{8}\right)f(t) = 0$$

and we take $L(q)$ to be (5) which is a Hecke eigenform in $M_3(\Gamma_0(4), (\frac{-4}{d}))$. We then have

$$f(q) \frac{q \frac{dt}{dq}}{t} = L(q^2).$$

□

Proof of Corollary 1.2. Since $\binom{p^r}{k} \equiv 0 \pmod{p^r}$ for $k \not\equiv 0 \pmod{p}$ and $r \geq 1$,

$$(23) \quad A(p^2) \equiv 2 \pmod{p^2}.$$

By Theorem 1.1 (with $m = 1$ and $r = 2$) and the fact that (4) is a weight 3 normalized Hecke eigenform, then (12) and (23) imply (9). Equation (9) also follows by taking $k = 3$ and $n = 1$ in Theorem 4.2 of [8]. A similar argument yields (10) since

$$C(p^2) \equiv 3 \pmod{p^2}$$

and (5) is a weight 3 normalized Hecke eigenform. Finally, we claim that

$$(24) \quad E(p) \equiv 4 \pmod{p^2}.$$

Since

$$4^{-2k} \binom{2k}{k}^2 = \left(\frac{-1}{k}\right)^2,$$

$$4^{p-1} \equiv 1 + \sum_{k=1}^{p-1} \binom{p}{k} \pmod{p^2},$$

$$\binom{p}{k} \equiv p \frac{(-1)^{k-1}}{k} \pmod{p^2},$$

and for $1 \leq k \leq \frac{p-1}{2}$,

$$\binom{p}{2k} \equiv \binom{p}{k} \frac{(-1)^k}{2} \pmod{p^2},$$

(24) is equivalent to

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \left[2(-1)^{k-1} - \frac{1}{2} \left(\frac{-1/2}{k}\right)^2 \right] \equiv 0 \pmod{p}.$$

Congruence (11) then follows by Proposition 2.2 and the fact that (5) is a weight 3 normalized Hecke eigenform.

□

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